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Some Applications to Variational–Hemivariational Inequalities of the Principle of Symmetric Criticality for Motreanu–Panagiotopoulos Type Functionals

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Abstract. Using the principle of symmetric criticality for Motreanu–Panagiotopoulos type functionals we give some existence and multiplicity results for a class of variational–hemivariational inequalities on $\mathbb{R}^{L+M}$.

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1. Introduction

Let $F: \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which is locally Lipschitz in the second variable (the real variable) and satisfies the following conditions:

(F1) $F(z,0)=0$ for all $z \in \mathbb{R}^L \times \mathbb{R}^M$ and there exist $c_1 > 0$ and $r \in]p, p^\star[$ such that

$$|\xi| \le c_1(|s|^{p-1} + |s|^{r-1}), \ \ \forall \xi \in \partial F(z,s), \ \ (z,s) \in \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R}.$$

We denoted by $\partial F(z,s)$ the generalized gradient of $F(z,\cdot)$ at the point $s \in \mathbb{R}$ and $p^\star = (L+M)p/L+M-p$ is the critical Sobolev exponent.

Let $a: \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$ $(L \ge 2)$ be a nonnegative continuous function satisfying the following assumptions:

(A_1) $a(x,y) \ge a_0 > 0$ if $|(x,y)| \ge R$ for a large $R > 0$;
(A_2) $a(x,y) \to +\infty$, when $|y| \to +\infty$ uniformly for $x \in \mathbb{R}^L$;
(A_3) $a(x,y) = a(x',y)$ for all $x, x' \in \mathbb{R}^L$ with $|x| = |x'|$ and all $y \in \mathbb{R}^M$.

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Consider the following subspaces of $W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M)$

$$\tilde{E}=\{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M): u(x,y)=u(x',y) \forall\ x, x' \in \mathbb{R}^L, |x|=|x'|, \forall y \in \mathbb{R}^M\},$$
$$E=\left\{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M): \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\right\},$$
$$E_a=\tilde{E} \cap E=\left\{u \in \tilde{E}: \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\right\}$$

endowed with the norm

$$\|u\|^p=\int_{\mathbb{R}^{L+M}} |\nabla u(z)|^p dz+\int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz$$

and the closed convex cone $\mathcal{K}=\{v \in E: v \geq 0$ a.e. in $\mathbb{R}^L \times \mathbb{R}^M\}$.

The aim of the present paper is to study the following eigenvalue problem (P_λ): *For* $\lambda > 0$ *find* $u \in \mathcal{K}$ *such that*

$$\int_{\mathbb{R}^{L+M}} |\nabla u(z)|^{p-2} \nabla u(z)(\nabla v(z)-\nabla u(z))dz+\int_{\mathbb{R}^{L+M}} a(z)u^{p-1}(z)(v(z)-u(z))dz$$
$$+\lambda \int_{\mathbb{R}^{L+M}} F^0(z,u(z);v(z)-u(z))dz \geq 0$$

for all $v \in \mathcal{K}$, where $F^0(z,s;t)$ is the generalized directional derivative of $F(z,\cdot)$ at the point s in the direction t.

The motivation to study this problem comes from some mechanical problems where a certain nondifferentiable term perturbs the classical functions. Panagiotopoulos [26] developed a more realistic approach, the so-called *theory of variational–hemivariational inequalities*, see for example the monographs Motreanu–Panagiotopoulos [20], Motreanu–Rădulescu [21] and Naniewicz–Panagiotopoulos [23], Gasiński–Papageorgiou [11], where the problems are studied on bounded domains.

On unbounded domains the methods must be changed, because the embedding of the Sobolev space $W^{1,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is not compact. A widely used tool in treating variational–hemivariational problems is the Principle of Symmetric Criticality, which states that it is enough to study the existence of critical points of a given function on a certain subspace, not on the whole space. For instance we mention the space of radially symmetric functions of $H^1(\mathbb{R}^N)$. In the differentiable case this principle was proved by R. S. Palais [25] and it was successfully applied by T. Bartsch and M. Willem in [5, 6]. The case of locally Lipschitz functions was developed by W. Krawcewicz and W. Marzantowicz [13] and applied by A. Kristály [16, 17], and also by C. Varga [30], Zs. Dályai and Cs. Varga [10]. Extensions of this principle for Szulkin [29] type functionals can be found in the paper [14] of J. Kobayashi and M. Ôtani. The case of Motreanu–Panagiotopoulos type functionals was investigated by A. Kristály, Cs. Varga, V. Varga in [18].

In the case when F is of class $\mathcal{C}^1$ with $F' = f$ problem (P_λ) becomes

$$(P_\lambda^1) \quad -\Delta_p u + a(x,y)u^{p-1} = \lambda f(x,y,u), \;\; y \in \mathbb{R}^L \times \mathbb{R}^M$$

and was studied by D. C. de Morais Filho, M. A. S. Souto and J. Marcos Do [22].

When $p=2, L=0$, and F is of class $\mathcal{C}^1$ with $F' = f$, then problem (P_λ) becomes

$$(P_\lambda^2) \quad -\Delta u + a(y)u = \lambda f(y,u), \;\; y \in \mathbb{R}^M.$$

When $a \equiv 1$ or a is radially symmetric or its level sets have some local or global properties, the existence and multiplicity of solutions of these problems were studied by Bartsch and Willem [5], T. Bartsch, Z. Liu, T. Weth [2, 3], T. Bartsch and Z.-Q. Wang [4], M. Willem [31].

If $p=2$, a is coercive and F is locally Lipschitz the problem (P_λ) was studied by F. Gazzola, V. Rădulescu in [12], while the case $p=2$, $a \equiv 1$ and F locally Lipschitz the problem (P_λ) was investigated by A. Kristály [16], Cs. Varga [30]. In the above mentioned papers $\mathcal{K}$ coincides with the whole space.

Here the main results (Theorems 3.1 and 3.2) establish the existence and multiplicity of solutions of (P_λ), by using the Principle of Symmetric Criticality (see [18]) in connection with the Mountain Pass Theorem (Corollary 3.2 from [20]) and a three critical point Theorem due to S. Marano and D. Motreanu (see Theorem B in [19]). To do this we also used the following embedding property given in [22] by D. C. de Morais Filho, M. A. S. Souto, J. Marcos Do: *E_a is continuously embedded in $L^s(\mathbb{R}^L \times \mathbb{R}^M)$ if $p \le s \le p^*$, and compactly embedded if $p < s < p^*$.* These results are given in Section 2 together with two examples. Section 3 and 4 contain the proofs of the main theorems together with some auxiliary results. The Appendix is devoted to the Principle of Symmetric Criticality for Motreanu–Panagiotopoulos functionals.

2. Basic Notions and Preliminary Results

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $h: X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \le L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of h at the point $u \in X$ in the direction $z \in X$ is

$$h^0(u;z) = \limsup_{w \to u,\, t \to 0^+} \frac{h(w+tz) - h(w)}{t}.$$

The *generalized gradient* of h at $u \in X$ is defined by

$$\partial h(u) = \{x^* \in X^* : \langle x^*, x\rangle \leq h^0(u; x), \ \forall x \in X\},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X.

Let $\mathcal{I} = h + \psi$, with $h: X \to \mathbb{R}$ locally Lipschitz and $\psi: X \to (-\infty, +\infty]$ convex, proper (i.e., $\psi \not\equiv +\infty$), and lower semicontinuous. $\mathcal{I}$ is a *Motreanu–Panagiotopoulos type* functional, see [20, Chapter 3].

DEFINITION 2.1 ([20, Definition 3.1]). *An element $u \in X$ is said to be a* critical point of $\mathcal{I} = h + \psi$, *if*

$$h^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X.$$

In this case, $\mathcal{I}(u)$ is a critical value of $\mathcal{I}$.

Let G be a topological group which acts *linearly* on X, i.e., the action $G \times X \to X$: $[g, u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear. The group G induces an action of the same type on the dual space X^* defined by $\langle gx^*, u\rangle = \langle x^*, g^{-1}u\rangle$ for every $g \in G, u \in X$ and $x^* \in X^*$. A function $h: X \to \mathbb{R} \cup \{+\infty\}$ is *G-invariant* if $h(gu) = h(u)$ for every $g \in G$ and $u \in X$. A set $K \subseteq X$ (or $K \subseteq X^*$) is *G-invariant* if $gK = \{gu : u \in K\} \subseteq K$ for every $g \in G$. Let

$$\Sigma = \{u \in X : gu = u \text{ for every } g \in G\}$$

the fixed point set of X under G. The Principle of Symmetric Criticality for Motreanu–Panagiotopoulos functionals is the following.

THEOREM 2.1. *Let X be a reflexive Banach space and $\mathcal{I} = h + \psi: X \to \mathbb{R} \cup \{+\infty\}$ be a Motreanu–Panagiotopoulos type functional. If a compact group G acts linearly on X, and the functionals h and ψ are G-invariant, then every critical point of $\mathcal{I}|_\Sigma$ is also a critical point of $\mathcal{I}$.*

DEFINITION 2.2 ([20, Definition 3.2]). *The functional $\mathcal{I} = h + \psi$ is said to* satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ (*shortly*, $(PS)_c$), *if every sequence (u_n) from X satisfying $\mathcal{I}(u_n) \to c$ and*

$$h^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

for a sequence (ε_n) in $[0, \infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, $\mathcal{I}$ is said to satisfy the Palais-Smale condition (*shortly*,(PS)).

For the later use a non smooth version of the Mountain Pass Theorem, i.e. Corollary 3.2 from [20].

THEOREM 2.2. *Assume that the functional* $I: X \to (-\infty, +\infty]$ *defined by* $I = h + \psi$, *satisfies* (PS), $I(0) = 0$, *and*

(i) *there exist constants* $\alpha > 0$ *and* $\rho > 0$, *such that* $I(u) \geq \alpha$ *for all* $||u|| = \rho$;
(ii) *there exists* $e \in X$, *with* $||e|| > \rho$ *and* $I(e) \leq 0$.

Then, the number

$$c = \inf_{f \in \Gamma} \sup_{t \in [0,1]} I(f(t)),$$

where

$$\Gamma = \{f \in C([0, 1], X): f(0) = 0, \quad f(1) = e\},$$

is a critical value of I *with* $c \geq \alpha$.

Let $h_1, h_2: X \to \mathbb{R}$ be locally Lipschitz functions, and let $\psi_1: X \to]-\infty, +\infty]$ be a convex, proper, lower semicontinuous function. Then the function $h_1 + \psi_1 + \lambda h_2$ is a Motreanu–Panagiotopoulos type functional for every $\lambda \in \mathbb{R}$. The following result was proved by Marano and Motreanu [19], Theorem B.

THEOREM 2.3. *Suppose that* $(X, ||\cdot||)$ *is a separable and reflexive Banach space. Let* $I_1 = h_1 + \psi_1$, $I_2 = h_2$, *and let* $\Lambda \subseteq \mathbb{R}$ *be an interval. We assume that:*

(a_1) h_1 *is weakly sequentially lower semicontinuous and* h_2 *is weakly sequentially continuous;*
(a_2) *for every* $\lambda \in \Lambda$ *the function* $I_1 + \lambda I_2$ *fulfils* $(PS)_c, c \in \mathbb{R}$, *and*

$$\lim_{||u|| \to +\infty} (I_1(u) + \lambda I_2(u)) = +\infty;$$

(a_3) *there exists a continuous concave function* $h: \Lambda \to \mathbb{R}$ *satisfying*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} (I_1(u) + \lambda I_2(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in \Lambda} (I_1(u) + \lambda I_2(u) + h(\lambda)).$$

Then, there exists an open interval $\Lambda_0 \subset \Lambda$, *such that for each* $\lambda \in \Lambda_0$ *the function* $I_1 + \lambda I_2$ *has at least three critical points in* X.

We introduce the functional $\mathcal{F}: E \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^{L+M}} F(z, u(z))dz.$$

In our proofs we will need the following result (see [10]).

LEMMA 2.1. *If the function F satisfies the condition* (F1)*, then $\mathcal{F}$ is locally Lipschitz and we have*

$$\mathcal{F}^0(u, v) \leq \int_{\mathbb{R}^{L+M}} F^0(z, u(z); v(z))dz,$$

for every $u, v \in E$. Moreover, the above inequality remains true on every closed subspace Y of E:

$$\left(\mathcal{F}\Big|_Y\right)^0 (u, v) \leq \int_{\mathbb{R}^{L+M}} F^0(z, u(z); v(z))dz,$$

for every $u, v \in Y$.

Let $\mathcal{I}_\lambda: E \to]-\infty, +\infty]$ be defined by

$$\mathcal{I}_\lambda(u) = \frac{1}{p}||u||^p - \lambda\mathcal{F}(u) + \psi_{\mathcal{K}}(u),$$

where $\psi_{\mathcal{K}}(u)$ denotes the indicator function of the closed convex cone $\mathcal{K}$, i.e.

$$\psi_{\mathcal{K}}(u) = \begin{cases} 0, & \text{if } x \in \mathcal{K}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly $\psi_{\mathcal{K}}$ is convex and lower-semicontinuous on E.

Now we rewrite problem (P_λ) by using the duality map. By Theorem 3.5 from [1] it follows that E is a separable, reflexive and uniform convex Banach space. We denote by $E^\star$ its dual. Let $J: E \to E^\star$ the duality mapping corresponding to the weight function $\varphi: [0, +\infty[\to [0, +\infty[$ defined by $\varphi(t) = t^{p-1}$, where $p \in]1, +\infty[$. It is well known that the duality mapping J satisfies the following conditions:

$$||Ju||_\star = \varphi(||u||) \quad \text{and} \quad \langle Ju, u\rangle = ||Ju||_\star ||u|| \quad \text{for all} \quad u \in E.$$

Moreover, the functional $\chi: E \to \mathbb{R}$ defined by $\chi(u) = (1/p)||u||^p$ is convex and Gateaux differentiable on E, and $d\chi = J$. For these properties of the duality mapping J we refer to [8].

The problem (P_λ) can be reformulated in the following way: *For* $\lambda > 0$ *find* $u \in \mathcal{K}$ *such that*

$$\langle Ju, v-u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); v(z)-u(z))dx \geq 0$$

for every $v \in \mathcal{K}$.

LEMMA 2.2. *Fix* $\lambda > 0$ *arbitrary. Every critical point* $u \in E$ *of the functional* $\mathcal{I}_\lambda$ *is a solution of the problem* (P_λ).

Proof. Since $u \in E$ is a critical point of the functional $\mathcal{I}_\lambda$, one has

$$\langle Ju, v-u\rangle + \lambda(-\mathcal{F})^0(u; v-u) + \psi_{\mathcal{K}}(v) - \psi_{\mathcal{K}}(u) \geq 0$$

for every $v \in E$. From Lemma 2.1 we obtain

$$\langle Ju, v-u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); u(z)-v(z))dz + \psi_{\mathcal{K}}(v) - \psi_{\mathcal{K}}(u) \geq 0$$

for every $v \in E$.

Therefore $u \in \mathcal{K}$ and for every $v \in \mathcal{K}$ we have

$$\langle Ju, v-u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); u(z)-v(z))dz \geq 0. \qquad \square$$

We consider a non-negative continuous function a: $\mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$ $(L \geq 2)$ satisfying the assumptions (A_1), (A_2), (A_3) given in Section 1 and recall the following subspaces of $W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M)$

$$\tilde{E} = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M): u(x, y) = u(x', y)\ \forall x, x' \in \mathbb{R}^L, |x| = |x'|, \forall y \in \mathbb{R}^M\},$$
$$E = \left\{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M): \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\right\},$$
$$E_a = \tilde{E} \cap E = \left\{u \in \tilde{E}: \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\right\}$$

endowed with the norm

$$\|u\|^p = \int_{\mathbb{R}^{L+M}} |\nabla u(z)|^p dz + \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz.$$

The next result is proved by de Morais Filho, Souto, Marcos Do [22] and is a very useful tool in our investigations.

THEOREM 2.4. *If* (A_1), (A_2) *and* (A_3) *hold, then the Banach space* E_a *is continuously embedded in* $L^s(\mathbb{R}^L \times \mathbb{R}^M)$, *if* $p \leq s \leq p^*$, *and compactly embedded if* $p < s < p^*$.

We have,

$$\|u\|_s \le C(s)\|u\| \quad \text{for each } u \in E_a,$$

where $\|\cdot\|_s$ is the norm in $L^s(\mathbb{R}^L \times \mathbb{R}^M)$ and $C(s) > 0$ is the embedding constant.

3. Main Results and Examples

Let

$$G = \left\{ g: E \to E: g(v) = v \circ \begin{pmatrix} R & 0 \\ 0 & Id_{\mathbb{R}^M} \end{pmatrix}, R \in O(\mathbb{R}^L) \right\},$$

where $O(\mathbb{R}^L)$ is the set of all rotations on $\mathbb{R}^L$ and $Id_{\mathbb{R}^M}$ denotes the $M \times M$ identity matrix. The elements of G leave $\mathbb{R}^{L+M}$ invariant, i.e. $g(\mathbb{R}^{L+M}) = \mathbb{R}^{L+M}$ for all $g \in G$.

The action of G over E is defined by

$$(gu)(z) = u(g^{-1}z), \quad g \in G, \quad u \in E, \quad \text{a.e. } z \in \mathbb{R}^{L+M}.$$

As usual we shall write gu in place of $\pi(g)u$.

A function u defined on $\mathbb{R}^{L+M}$ is said to be G-invariant, if

$$u(gz) = u(z), \ \forall g \in G, \ \text{a.e. } z \in \mathbb{R}^{L+M}.$$

Then $u \in E$ is G-invariant if and only if $u \in \Sigma$, where

$$\Sigma := E_a = \tilde{E} \cap E.$$

We observe that the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^{L+M}} (|\nabla u(z)|^p + a(z)|u(z)|^p) dz \right\}^{\frac{1}{p}}$$

is G-invariant.

In order to study our problem we give the assumptions on the nonlinear function F. We assume that $F: \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the second variable (the real variable) and satisfies the following conditions:

(F1) $F(z, 0) = 0$, and there exist $c_1 > 0$ and $r \in]p, p^\star[$ such that

$$|\xi| \le c_1(|s|^{p-1} + |s|^{r-1}), \ \forall \xi \in \partial F(z, s), \ (z, s) \in \mathbb{R}^{L+M} \times \mathbb{R}.$$

We denoted by $\partial F(z,s)$ the generalized gradient of $F(z,\cdot)$ at the point $s\in\mathbb{R}$.

(F2) $\lim\limits_{s\to 0}\dfrac{\max\{|\xi|:\xi\in\partial F(z,s)\}}{|s|^{p-1}}=0$ uniformly for every $z\in\mathbb{R}^{L+M}$.

(F3) There exists $\nu>p$ such that

$$\nu F(z,s)+F^0(z,s;-s)\leq 0,\ \ \forall(z,s)\in\mathbb{R}^{L+M}\times\mathbb{R}.$$

(F4) There exists $r>0$ such that

$$\inf\{F(z,s):(z,|s|)\in\mathbb{R}^{L+M}\times[r,\infty)\}>0.$$

Arguing as in the proof of Lemma 4.1 in [17] one has.

Remark 3.1. (a) If $F:\mathbb{R}^{L+M}\times\mathbb{R}\to\mathbb{R}$ satisfies *(F1)* and *(F2)*, then for every $\varepsilon>0$, there exists $c(\varepsilon)>0$ such that

(i) $|\xi|\leq\varepsilon|s|^{p-1}+c(\varepsilon)|s|^{r-1},\ \ \forall\xi\in\partial F(z,s),\ \ (z,s)\in\mathbb{R}^{L+M}\times\mathbb{R};$
(ii) $|F(z,s)|\leq\varepsilon|s|^{p}+c(\varepsilon)|s|^{r},\ \ \forall(z,s)\in\mathbb{R}^{L+M}\times\mathbb{R}.$

(b) If $F:\mathbb{R}^{L+M}\times\mathbb{R}\to\mathbb{R}$ satisfies *(F1)*, *(F3)* and *(F4)*, then there exist $c_2,c_3>0$ and $\nu\in]p,p^\star[$ such that

$$F(z,s)\geq c_2|s|^{\nu}-c_3|s|^{p}.$$

To study the existence of the solutions of problem (P_λ), it is sufficient to prove the existence of critical points of the functional $\mathcal{I}_\lambda$ (see Lemma 2.2).
The **main results of the paper** are:

THEOREM 3.1. *Let* $F:\mathbb{R}^L\times\mathbb{R}^M\times\mathbb{R}\to\mathbb{R}$ *be a function, which satisfies* (F1)–(F4) *and* $F(\cdot,s)$ *is G-invariant for every* $s\in\mathbb{R}$. *Then for every* $\lambda>0$ *problem* (P_λ) *has a nontrivial positive solution.*

From the other hand, by replacing (F3) and (F4) with the following two conditions

(F′3) There exist $q\in]0,p[,\nu\in[p,p^\star],\alpha\in L^{\frac{\nu}{\nu-q}}(\mathbb{R}^{L+M}),\beta\in L^1(\mathbb{R}^{L+M})$ such that

$$F(z,s) \leq \alpha(z)|s|^q + \beta(z)$$

for all $s \in \mathbb{R}$ and a.e. $z \in \mathbb{R}^{L+M}$;

(F′4) There exists $u_0 \in \mathcal{K}$ such that $\displaystyle\int_{\mathbb{R}^{L+M}} F(z, u_0(z))dz > 0$;

we obtain at least three solutions to problem (P_λ). To be precise we establish the following theorem.

THEOREM 3.2. *Let* $F: \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ *be a function which satisfies* (F1), (F2), (F′3), (F′4) *and* $F(\cdot, s)$ *is G-invariant for all* $s \in \mathbb{R}$. *Then there exists an open interval* $\Lambda_0 \subset \Lambda$ *such that for each* $\lambda \in \Lambda_0$ *problem* (P_λ) *has at least three distinct solutions which are axially symmetric.*

Remark 3.2. If in the above theorem we change the condition (F′3) with

(F″3) $\limsup_{|s|\to+\infty} \dfrac{F(z,s)}{|s|^p} \leq 0$, *uniformly* in $z \in \mathbb{R}^L \times \mathbb{R}^M$,

then the conclusion of Theorem 3.2 remains true.

Here we give two examples, where the above results can be applied successfully.

EXAMPLE 3.1. Let $k \in \mathbb{R}, k > 1$. We define the sequence of real numbers (A_n) by $A_0 = 0$, and

$$A_n = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k}, \quad n \geq 1.$$

Let $r > p > 2$. We consider the functions $f, F: \mathbb{R} \to \mathbb{R}$ given respectively by

$$f(s) = s|s|^{p-2}(|s|^{r-p} + A_n) \quad \text{for } s \in]-n-1, -n] \cup [n, n+1[, \ n \in \mathbb{N},$$
$$F(u) = \int_0^u f(s)ds \quad \text{for } u \in]-n-1, -n] \cup [n, n+1[, \ n \in \mathbb{N}.$$

Clearly F satisfies (F1), (F2), (F3) and (F4), hence owing to Theorem 3.1 problem (P_λ) has a nontrivial positive solution.

EXAMPLE 3.2. Let $A: \mathbb{R}^L \to \mathbb{R}$ be a continuous, nonnegative, not identically zero, axially symmetric function with compact support in $\mathbb{R}^L$. We consider $F: \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ defined by

$$F((x,y),s) = A(x)\min\{s^r, |s|^q\} \quad \text{for } (x,y) \in \mathbb{R}^L \times \mathbb{R}^M, \ s \in \mathbb{R},$$

where $r \in]p, (L+M)p/L+M-p[$ is an odd number and $q \in]0, p[$. The function F satisfies the assumptions (F1), (F2), (F′3) and (F′4) and $F(\cdot, s)$

is G-invariant for all $s \in \mathbb{R}$. Theorem 3.2 implies that there exists an open interval $\Lambda_0 \subset \Lambda$ such that for each $\lambda \in \Lambda_0$ problem (P_λ) has at least three distinct solutions which are axially symmetric.

4. Proof of Theorem 3.1

Because the cone $\mathcal{K}$ is G-invariant, it follows that $\psi_{\mathcal{K}}$ is G-invariant. Taking into account that the action of G is linear and isometric on E, we deduce that the function $\chi(u) = \frac{1}{p}||u||^p$ is G-invariant. The function $\mathcal{F}$ is also G-invariant, because $F(\cdot, s)$ is G-invariant for every $s \in \mathbb{R}$. If we apply Theorem 2.1, it is sufficient to prove that the functional $\mathcal{I}_\Sigma := \mathcal{I}_\lambda|_\Sigma$ has critical points, which implies that the functional $\mathcal{I}_\lambda$ has critical points, which are solutions for problem (P_λ). We introduce the following notations:

$$||\cdot||_\Sigma = ||\cdot||\Big|_\Sigma, \quad \mathcal{F}_\Sigma = \mathcal{F}\Big|_\Sigma, \quad \psi_\Sigma = \psi_{\mathcal{K}}\Big|_\Sigma$$

and the restricted duality map $J_\Sigma : \Sigma \to \Sigma^*$ with $J_\Sigma = J\Big|_\Sigma$. Therefore we have

$$\mathcal{I}_\Sigma(u) = \frac{1}{p}||u||_\Sigma^p - \lambda \mathcal{F}_\Sigma(u) + \psi_\Sigma(u).$$

In the next we verify that the conditions of Theorem 2.2 are satisfied by the functional $\mathcal{I}_\Sigma$.

PROPOSITION 4.1. *If $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies the conditions* (F1)–(F3) *and $F(\cdot, s), s \in \mathbb{R}$ is G- invariant, then $\mathcal{I}_\Sigma$ satisfies the (PS) condition, for every $\lambda > 0$.*

Proof. Let $\lambda > 0$ and $c \in \mathbb{R}$ be some fixed numbers and let $(u_n) \subset \Sigma$ be a sequence such that

$$\mathcal{I}_\Sigma(u_n) = \frac{1}{p}||u_n||_\Sigma^p - \lambda \mathcal{F}_\Sigma(u_n) + \psi_\Sigma(u_n) \to c \tag{4.1}$$

and for every $v \in \Sigma$ we have

$$\begin{aligned} &\langle J_\Sigma u_n, v - u_n \rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - v(z)) dz + \\ &\quad + \psi_\Sigma(v) - \psi_\Sigma(u_n) \geq -\varepsilon_n ||v - u_n||_\Sigma, \end{aligned} \tag{4.2}$$

for a sequence (ε_n) in $[0, +\infty[$ with $\varepsilon_n \to 0$.

By (4.1) one concludes that $(u_n) \subset \mathcal{K} \cap \Sigma$. Setting $v = 2u_n$ in (4.2), we obtain

$$\langle J_\Sigma u_n, u_n \rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); -u_n(z))dz \geq -\varepsilon_n ||u_n||_\Sigma. \tag{4.3}$$

By (4.1) one has for large $n \in \mathbb{N}$ that

$$c + 1 \geq \frac{1}{p} ||u_n||_\Sigma^p - \lambda \mathcal{F}_\Sigma(u_n). \tag{4.4}$$

We multiply inequality (4.3) with ν^{-1} and use Lemma 2.1 to obtain

$$\varepsilon_n \frac{||u_n||_\Sigma}{\nu} \geq -\frac{\langle J_\Sigma u_n, u_n \rangle}{\nu} - \frac{\lambda}{\nu} \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); -u_n(z))dz. \tag{4.5}$$

Adding the inequalities (4.4) and (4.5), and using (F3) we get

$$\begin{aligned} c + 1 + \frac{\varepsilon_n}{\nu} ||u_n||_\Sigma &\geq \left(\frac{1}{p} - \frac{1}{\nu}\right) ||u_n||_\Sigma^p - \lambda \int_{\mathbb{R}^{L+M}} [F(z, u_n(z)) + \\ &\qquad + \frac{1}{\nu} F^0(z, u_n(z); -u_n(z))]dz \\ &\geq \left(\frac{1}{p} - \frac{1}{\nu}\right) ||u_n||_\Sigma^p. \end{aligned}$$

From this, we get that the sequence $(u_n) \subset \mathcal{K} \cap \Sigma$ is bounded. Because E is reflexive, it follows that Σ is reflexive too and there exists an element $u \in \Sigma$ such that $u_n \rightharpoonup u$ weakly. Since $\mathcal{K} \cap \Sigma$ is closed and convex, we get $u \in \mathcal{K} \cap \Sigma$. Moreover, from (4.2) with $v = u$ we obtain

$$\langle J_\Sigma u_n, u - u_n \rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - u(z))dz \geq -\varepsilon_n ||u_n - u||_\Sigma. \tag{4.6}$$

From this we get

$$\begin{aligned} \langle J_\Sigma u_n, u_n - u \rangle &\leq \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - u(z))dz + \varepsilon_n ||u_n - u||_\Sigma \\ &\leq \lambda \int_{\mathbb{R}^{L+M}} \max\{\xi_n(z)(u_n(z) - u(z)) : \xi_n(z) \in \partial F(z, u_n(z))\}dz \\ &\quad + \varepsilon_n ||u_n - u||_\Sigma \\ &\leq \lambda \int_{\mathbb{R}^{L+M}} \left(\varepsilon |u_n(z)|^{p-1} + c(\varepsilon)|u_n(z)|^{r-1}\right) |u_n(z) - u(z)| dz \\ &\quad + \varepsilon_n ||u_n - u||_\Sigma. \end{aligned}$$

Hence, by Hölder's inequality and the fact that the inclusion $\Sigma \hookrightarrow L^p(\mathbb{R}^{L+M})$ is continuous (see Theorem 2.4), we obtain

$$\langle J_\Sigma u_n, u_n - u\rangle \le \lambda\varepsilon C(p)||u_n - u||_\Sigma ||u_n||_p^{p-1} + \\ +\lambda c(\varepsilon)||u_n - u||_r ||u_n||_r^{r-1} + \varepsilon_n ||u_n - u||_\Sigma.$$

Moreover, the inclusion $\Sigma \hookrightarrow L^r(\mathbb{R}^{L+M})$ is compact for $r \in]p, p^*[$ (see Theorem 2.4), therefore $||u_n - u||_r \to 0$ as $n \to +\infty$. For $\to 0^+$ and $n \to +\infty$ we obtain that $\limsup_{n\to+\infty}\langle J_\Sigma u_n, u_n - u\rangle \le 0$. Finally, since the duality operator J_Σ has the (S_+) property (see, Proposition 2.1 in [24]) we obtain $u_n \to u$ in $\mathcal{K}$, because $\mathcal{K}$ is closed. □

PROPOSITION 4.2. *If* $F: \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ *verifies* (F1)–(F4) *and* $F(\cdot, s)$ *is G-invariant for every* $s \in \mathbb{R}$*, then for every* $\lambda > 0$ *the following assertions are true:*

(i) *there exist constants* $\alpha_\lambda > 0$ *and* $\rho_\lambda > 0$ *such that* $\mathcal{I}_\Sigma(u) \ge \alpha_\lambda$ *for all* $||u||_\Sigma = \rho_\lambda$*;*
(ii) *there exists* $e_\lambda \in \mathcal{K}$ *with* $||e_\lambda|| > \rho_\lambda$ *and* $\mathcal{I}_\Sigma(e_\lambda) \le 0$*.*

Proof. From Remark 3.1 and from the fact that the embedding $\Sigma \hookrightarrow L^l(\mathbb{R}^{L+M})$ is continuous for $l \in [p, p^\star]$, it follows that

$$\mathcal{F}_\Sigma(u) \le \varepsilon C^p(p)||u||_\Sigma^p + c(\varepsilon)C^r(r)||u||_\Sigma^r,$$

for every $u \in \Sigma$. It is suffices to restrict our attention to elements u which belong to $\mathcal{K} \cap \Sigma$, otherwise $\mathcal{I}_\Sigma(u)$ will be $+\infty$, i.e. (i) holds trivially.

Let $\lambda > 0$ be arbitrary. We choose $\varepsilon \in]0, 1/p\lambda C^p(p)[$ and for $u \in \mathcal{K} \cap \Sigma$ we have

$$\mathcal{I}_\Sigma(u) = \frac{1}{p}||u||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u) \ge \left(\frac{1}{p} - \lambda\varepsilon C^p(p)\right)||u||_\Sigma^p - \lambda c(\varepsilon)C^r(r)||u||_\Sigma^r.$$

We denote by $A = \frac{1}{p} - \lambda\varepsilon C^p(p)$ and $B = \lambda c(\varepsilon)C^r(r)$ and we consider the function $g: \mathbb{R}_+ \to \mathbb{R}$ given by $g(t) = At^p - Bt^r$. The function g attains its global maximum in the point $t_\lambda = \left(\frac{pA}{rB}\right)^{\frac{1}{r-p}}$. If we take $\rho_\lambda = t_\lambda$ and $\alpha_\lambda \in]0, g(t_\lambda)[$, the condition (i) is fulfilled.

To prove (ii) from (b) Remark 3.1 we observe that for every $u \in \mathcal{K} \cap \Sigma$ we have

$$\mathcal{I}_\Sigma(u) \le \frac{1}{p}||u||_\Sigma^p + \lambda c_3 C^p(p)||u||_\Sigma^p - \lambda c_2||u||_\nu^\nu.$$

If we fix an element $v \in (\mathcal{K} \cap \Sigma)\setminus\{0\}$ and in place of u we put tv, then we have

$$\mathcal{I}_\Sigma(tv) \leq \left(\frac{1}{p} + \lambda c_3 C^p(p)\right) ||v||_\Sigma^p t^p - \lambda c_2 ||v||_\nu^\nu t^\nu.$$

From this we see that if t is large enough, then $||tv||_\Sigma > \rho_\lambda$ and $\mathcal{I}_\Sigma(tv) < 0$. If we take $e_\lambda = tv$ we obtain the desired results. □

The Proof of Theorem 3.1. Now we prove that the conditions of Theorem 2.2 are satisfied by the functional $\mathcal{I}_\Sigma$. Because $F(z, 0) = 0$, it follows that

$$\mathcal{I}_\Sigma(0) = \int_{\mathbb{R}^{L+M}} F(z, 0)dz = 0.$$

From Proposition 4.1 we get that $\mathcal{I}_\Sigma$ satisfies the (PS) condition. Proposition 4.2 implies that $\mathcal{I}_\Sigma$ satisfies the conditions (i) and (ii) from Theorem 2.2, hence the number

$$c_\lambda = \inf_{f \in \Gamma} \sup_{t \in [0,1]} I_\Sigma(f(t)),$$

where

$$\Gamma_\lambda = \{f \in C([0, 1], \Sigma) : f(0) = 0, \quad f(1) = e_\lambda\},$$

is a critical value of $\mathcal{I}_\Sigma$ with $c_\lambda \geq \alpha_\lambda$. □

5. Proof of Theorem 3.2

Now we give some auxiliary results in order to prove Theorem 3.2. We consider the functional $f: E \times \Lambda \to]-\infty, +\infty]$ given by $f(u, \lambda) = I_1(u) + \lambda I_2(u)$, where

$$I_1(u) = \frac{1}{p}||u||^p + \psi_\mathcal{K}(u), \quad I_2(u) = -\mathcal{F}(u) = -\int_{\mathbb{R}^{L+M}} F(z, u(z))dz.$$

As in Lemma 2.2 we have that every critical point of the function $f = I_1 + \lambda I_2$ is a solution of problem (P_λ). Using Theorem 2.1 it is sufficient to prove that the functional $f_\Sigma = (I_1 + \lambda I_2)|_\Sigma$ satisfies conditions from Theorem 2.3, where we choose $h_1, \Psi_1, h_2: \Sigma \to \mathbb{R}$

$$h_1(u) = \frac{1}{p}||u||_\Sigma^p, \quad \Psi_1(u) = \psi_\Sigma(u),$$

$$h_2(u) = -\mathcal{F}_\Sigma(u) = -\int_{\mathbb{R}^{L+M}} F(z, u(z))dz, \quad u \in \Sigma,$$

and take

$$I_1 = h_1 + \Psi_1, \;\; I_2 = h_2.$$

First we prove that (a_1) holds.

PROPOSITION 5.1. *If* $F: \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ *verifies the conditions* (F1) *and* (F2), *then* h_1 *is weakly sequentially lower semicontinuous and* h_2 *is weakly sequentially continuous.*

Proof. The weakly sequentially lower semicontinuity of $h_1 = 1/p||\cdot||^p_\Sigma$ is standard (every convex lower semicontinuous function is sequentially lower semicontinuous, see e.g. [7]).

In order to prove the weakly sequentially continuity of h_2 we assume that (u_n) is a sequence in Σ such that $u_n \rightharpoonup u$ (in Σ). We will prove that $\mathcal{F}_\Sigma(u_n) \to \mathcal{F}_\Sigma(u)$.

By Lebourg's Mean Value Theorem (see [9]) it follows that there exist $\theta_n \in [0, 1]$ and $v_n \in \partial\mathcal{F}_\Sigma(u + \theta_n(u_n - u))$ such that

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) = \langle v_n, u_n - u\rangle.$$

We denote $w_n = u + \theta_n(u_n - u)$. Using the definition of $\mathcal{F}^0_\Sigma$, Lemma 2.1 it follows that

$$\begin{aligned}\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) &\le (\mathcal{F}_\Sigma)^0(w_n; u_n - u) \le \int_{\mathbb{R}^{L+M}} F^\circ(z, w_n(z); u_n(z) - u(z))dz \\ &= \int_{\mathbb{R}^{L+M}} \max\Big\{\langle v(z), u_n(z) - u(z)\rangle : v \in \partial F(z, w_n(z))\Big\}.\end{aligned}$$

Now we use Remark 3.1 to get

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) \le \int_{\mathbb{R}^{L+M}} \Big(\varepsilon|w_n(z)|^{p-1} + c(\varepsilon)|w_n(z)|^{r-1}\Big)|u_n(z) - u(z)|dz.$$

We use Hölder's inequality and the fact that the inclusion $\Sigma \hookrightarrow L^p(\mathbb{R}^{L+M})$ is continuous (see Theorem 2.4) to obtain

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) \le \varepsilon C(p)\|u_n - u\|_\Sigma \|w_n\|^{p-1}_p + c(\varepsilon)C(r)\|u_n - u\|_r \|w_n\|^{r-1}_r. \tag{5.1}$$

Now we use the same ideas as before for $-\mathcal{F}_\Sigma$ and find the existence of $\tau_n \in [0, 1]$ and $\hat{v}_n \in \partial(-\mathcal{F}_\Sigma)(u + \tau_n(u_n - u))$ such that

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) = \langle \hat{v}_n, u_n - u\rangle.$$

We denote $\hat{w}_n = u + \tau_n(u_n - u)$. Using the definition of $-\mathcal{F}_\Sigma^0$, and properties of the generalized gradient (see [9]), it follows that

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) \leq (-\mathcal{F}_\Sigma)^0(\hat{w}_n; u_n - u) = (\mathcal{F}_\Sigma)^0(\hat{w}_n; u - u_n).$$

Analogously to (5.1) we get

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) \leq \varepsilon C(p)\|u_n - u\|_\Sigma \|\hat{w}_n\|_p^{p-1} + c(\varepsilon)C(r) \times \\ \times \|u_n - u\|_r \|\hat{w}_n\|_r^{r-1}. \tag{5.2}$$

Using (5.1) and (5.2) we have

$$|\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u)| \leq \varepsilon C(p)\|u_n - u\|_\Sigma (\|w_n\|_p^{p-1} + \\ + \|\hat{w}_n\|_p^{p-1}) + c(\varepsilon)C(r)\|u_n - u\|_r (\|w_n\|_r^{r-1} + \|\hat{w}_n\|_r^{r-1}). \tag{5.3}$$

The inclusion $\Sigma \hookrightarrow L^r(\mathbb{R}^{L+M})$ is compact for $r \in]p, p^*[$ (see Theorem 2.4), then we get that $||u_n - u||_r \to 0$ as $n \to +\infty$, while the sequences (w_n) and $(\hat{w}_n)$ are bounded in the $\|\cdot\|_p$ and $\|\cdot\|_r$ norms. Then in (5.3) we get $\mathcal{F}_\Sigma(u_n) \to \mathcal{F}_\Sigma(u)$. Hence h_2 is weakly sequentially continuous. □

Proof of Theorem 3.2. For this let $u \in \mathcal{K} \cap \Sigma$, from condition $(F'3)$ and from the fact that the embedding $\Sigma \hookrightarrow L^\nu(\mathbb{R}^{L+M})$ is continuous and $q < p$ it follows that

$$\begin{aligned} f_\Sigma(u,\lambda) &\geq \frac{1}{p}||u||_\Sigma^p - \lambda \int_{\mathbb{R}^{L+M}} \alpha(z)|u(z)|^q dz - \lambda \int_{\mathbb{R}^{L+M}} \beta(z) dz \\ &\geq \frac{1}{p}||u||_\Sigma^p - \lambda ||\alpha||_{\frac{\nu}{\nu-q}} ||u||_\nu^q - \lambda||\beta||_1 \\ &\geq \frac{1}{p}||u||_\Sigma^p - \lambda ||\alpha||_{\frac{\nu}{\nu-q}} C^q(q)||u||_\Sigma^q - \lambda||\beta||_1. \end{aligned}$$

Therefore, if $||u||_\Sigma \to +\infty$, we have $f_\Sigma(u,\lambda) \to +\infty$. Let $(u_n) \subset \mathcal{K} \cap \Sigma$ be a sequence such that

$$f_\Sigma(u_n, \lambda) \to c \tag{5.4}$$

and for every $v \in \Sigma$ we have

$$\langle J_\Sigma u_n, v - u_n\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - v(z)) dz + \psi_\Sigma(v) - \psi_\Sigma(u_n) \\ \geq -\varepsilon_n ||v - u_n||_\Sigma, \tag{5.5}$$

for a sequence (ε_n) in $[0,+\infty[$ with $\varepsilon_n \to 0$. From (5.4) follows that the sequence (u_n) is bounded in $\mathcal{K}\cap\Sigma$ and as in Proposition 4.1 we get that there exists an element $u\in\mathcal{K}\cap\Sigma$ such that $u_n \to u$. Let us define the function

$$g(t)=\sup\left\{\mathcal{F}_\Sigma(u)\colon \frac{1}{p}||u||_\Sigma^p \le t\right\}.$$

Using (ii) from Remark 3.1 and the fact that the inclusion $\Sigma \hookrightarrow L^l(\mathbb{R}^{L+M})$, $l\in[p, p^\star]$ is continuous, it follows that

$$g(t)\le \varepsilon C^p(p)t + c(\varepsilon)C^r(r)t^{\frac{r}{p}}. \tag{5.6}$$

On the other hand $g(t)\ge 0$ for each $t>0$, then from the above relation we get

$$\lim_{t\to 0^+}\frac{g(t)}{t}=0. \tag{5.7}$$

By (F′4) it is clear that $u_0\neq 0$ (since $\mathcal{F}(0)=0$). Therefore it is possible to choose a number η such that

$$0<\eta<\mathcal{F}_\Sigma(u_0)\left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}.$$

From $\lim_{t\to 0^+} g(t)/t = 0$ it follows the existence of a number $t_0 \in]0, 1/p||u_0||_\Sigma^p[$ such that $g(t_0)<\eta t_0$. Thus

$$g(t_0)<\left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}\mathcal{F}_\Sigma(u_0)t_0.$$

Let $\rho_0>0$ such that

$$g(t_0)<\rho_0<\left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}\mathcal{F}_\Sigma(u_0)t_0. \tag{5.8}$$

Due to the choice of t_0 and (5.8) we have

$$\rho_0<\mathcal{F}_\Sigma(u_0). \tag{5.9}$$

Define $h\colon \Lambda=[0,+\infty[\to\mathbb{R}$ by $h(\lambda)=\rho_0\lambda$. We prove that the function h satisfies the inequality

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{K}\cap\Sigma}(f_\Sigma(u,\lambda)+h(\lambda))<\inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}(f_\Sigma(u,\lambda)+h(\lambda)).$$

The function

$$\Lambda \ni \lambda \mapsto \inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_{\Sigma}^{p}+\lambda(\rho_0-\mathcal{F}_{\Sigma}(u))\right]$$

is obviously upper semicontinuous on Λ.

From (5.9) it follows that

$$\lim_{\lambda\to+\infty}\inf_{u\in\mathcal{K}\cap\Sigma}[f_{\Sigma}(u,\lambda)+\rho_0\lambda]\le\lim_{\lambda\to+\infty}\left[\frac{1}{p}||u_0||_{\Sigma}^{p}+\lambda(\rho_0-\mathcal{F}_{\Sigma}(u_0))\right]=-\infty. \tag{5.10}$$

Thus we find an element $\overline{\lambda}\in\Lambda$ such that

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{K}\cap\Sigma}(f_{\Sigma}(u,\lambda)+\rho_0\lambda)=\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_{\Sigma}^{p}+\overline{\lambda}(\rho_0-\mathcal{F}_{\Sigma}(u))\right]. \tag{5.11}$$

From $g(t_0)<\rho_0$ it follows that for all $u\in\Sigma$ with $1/p||u||_{\Sigma}^{p}\le t_0$, we have $\mathcal{F}_{\Sigma}(u)<\rho_0$. Hence

$$t_0\le\inf\left\{\frac{1}{p}||u||_{\Sigma}^{p}:\mathcal{F}_{\Sigma}(u)\ge\rho_0\right\}. \tag{5.12}$$

On the other hand,

$$\begin{aligned}\inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}(f_{\Sigma}(u,\lambda)+\rho_0\lambda)&=\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_{\Sigma}^{p}+\sup_{\lambda\in\Lambda}(\lambda(\rho_0-\mathcal{F}_{\Sigma}(u)))\right]\\&=\inf\left\{\frac{1}{p}||u||_{\Sigma}^{p}:\mathcal{F}_{\Sigma}(u)\ge\rho_0\right\}.\end{aligned}$$

Thus (5.12) is equivalent with

$$t_0\le\inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}[f_{\Sigma}(u,\lambda)+\rho_0\lambda]. \tag{5.13}$$

There are two distinct cases:

(I) If $0\le\overline{\lambda}<t_0/\rho_0$, we have

$$\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_{\Sigma}^{p}+\overline{\lambda}(\rho_0-\mathcal{F}_{\Sigma}(u))\right]\le f_{\Sigma}(0,\overline{\lambda})=\overline{\lambda}\rho_0<t_0.$$

Combining the above inequality with (5.11) and (5.13) we obtain the inequality from (a_2) Theorem 2.3.

(II) If $t_0/\rho_0 \leq \overline{\lambda}$, then from $\rho_0 < \mathcal{F}_\Sigma(u_0)$ and (5.8) it follows

$$\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p + \overline{\lambda}(\rho_0 - \mathcal{F}_\Sigma(u))\right] \leq \frac{1}{p}||u_0||_\Sigma^p + \overline{\lambda}(\rho_0 - \mathcal{F}_\Sigma(u_0))$$
$$\leq \frac{1}{p}||u_0||_\Sigma^p + \frac{t_0}{\rho_0}(\rho_0 - \mathcal{F}_\Sigma(u_0)) < t_0.$$

Theorem 2.3 implies that there exists an open interval $\Lambda_0 \subset \Lambda$, such that for each $\lambda \in \Lambda_0$, the function $f_\Sigma(\cdot,\lambda)$ has at least three critical points in $\mathcal{K}\cap\Sigma$. Therefore, problem (P_λ) has at least three distinct solutions for every $\lambda \in \Lambda_0$. This ends the proof of Theorem 3.2. □

Final remark. The results of this article remain true for a more general class of convex functions defined on the cone of positive functions, than the indicator function of $\mathcal{K}$. This will be investigated in a furthercoming paper.

6. Appendix–The Principle of Symmetric Criticality for Motreanu–Panagiotopolus functionals

Following the paper of A. Kristály, C. Varga, V. Varga from [18] we present in this section the Principle of Symmetric Criticality for Motreanu–Panagiotopolus functionals.

Let $\mathcal{I}$ be a *Motreanu–Panagiotopoulos type* functional, i.e. $\mathcal{I} = h + \psi$, with $h: X \to \mathbb{R}$ locally Lipschitz and $\psi: X \to (-\infty, +\infty]$ convex, proper (i.e. $\psi \not\equiv +\infty$), and lower semicontinuous functions.

One can characterize the critical points (in the sense of Definition 2.1) by means of differential inclusions.

PROPOSITION 6.1 ([15]). *An element $u \in X$ is a critical point of $\mathcal{I} = h + \psi$, if and only if $0 \in \partial h(u) + \partial\psi(u)$, where $\partial\psi(u)$ denotes the subdifferential of the convex function ψ at u, i.e.*

$$\partial\psi(u) = \{x^* \in X^*: \psi(v) - \psi(u) \geq \langle x^*, v-u\rangle_X \text{ for every } v \in X\}.$$

Let G be a topological group which acts *linearly* on X, i.e., the action $G \times X \to X: [g,u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear. The group G induces an action of the same type on the dual space X^* defined by $\langle gx^*, u\rangle_X = \langle x^*, g^{-1}u\rangle_X$ for every $g \in G$, $u \in X$ and $x^* \in X^*$. A function $h: X \to \mathbb{R}\cup\{+\infty\}$ is *G-invariant* if $h(gu) = h(u)$ for every $g \in G$ and $u \in X$. A set $K \subseteq X$ (or $K \subseteq X^*$) is *G-invariant* if $gK = \{gu: u \in K\} \subseteq K$ for every $g \in G$. Let

$$\Sigma = \{u \in X: gu = u \text{ for every } g \in G\}$$

be the fixed point set of X under G.

In order to give the proof of Theorem 2.1, we recall first some facts from [14]. Let

$$\Phi(X)=\{\psi:X\to\mathbb{R}\cup\{\infty\}:\psi \text{ is convex, proper, lower semicontinuous}\};$$
$$\Phi_G(X)=\{\psi\in\Phi(X):\ \psi \text{ is } G-\text{invariant}\};$$
$$\Gamma_G(X^*)=\{K\subseteq X^*:\ K \text{ is } G-\text{invariant, weak}^*-\text{closed, convex}\}.$$

PROPOSITION 6.2 ([14, Theorem 3.16]). *Assume that a compact group G acts linearly on a reflexiv Banach space X. Then for every $K\in\Gamma_G(X^*)$ and $\psi\in\Phi_G(X)$ one has*

$$K|_\Sigma\cap\partial(\psi|_\Sigma)(u)\neq\emptyset\Rightarrow K\cap\partial\psi(u)\neq\emptyset,\ \ u\in\Sigma, \tag{6.1}$$

where $K|_\Sigma=\{x^*|_\Sigma: x^*\in K\}$ with $\langle x^*|_\Sigma,u\rangle_\Sigma=\langle x^*,u\rangle_X,\ \ u\in\Sigma$.

Let $A:X\to X$ be the *averaging operator* over G, defined by

$$Au=\int_G gud\mu(g),\ \ u\in X, \tag{6.2}$$

where μ is the normalized Haar measure on G. Relation (6.2) means

$$\langle x^*,Au\rangle_X=\int_G\langle x^*,gu\rangle_X d\mu(g),\ \ u\in X,\ \ x^*\in X^*. \tag{6.3}$$

It is easy to verify that A is a continuous linear projection from X to Σ and for every G-invariant closed convex set $K\subseteq X$ we have $A(K)\subseteq K$. The adjoint operator $A^*:\Sigma^*\to X^*$ of $A:X\to\Sigma$ is defined by

$$\langle w^*,Az\rangle_\Sigma=\langle A^*w^*,z\rangle_X,\ \ z\in X,\ \ w^*\in\Sigma^*. \tag{6.4}$$

LEMMA 6.1 *Let $h:X\to\mathbb{R}$ be a G-invariant locally Lipschitz function and $u\in\Sigma$. Then*

(a) $\partial(h|_\Sigma)(u)\subseteq\partial h(u)|_\Sigma$.
(b) $\partial h(u)\in\Gamma_G(X^*)$.

Proof. (a) Let us fix $w^*\in\partial(h|_\Sigma)(u)$. Then by definition, one has

$$\langle w^*,v\rangle_\Sigma\le(h|_\Sigma)^0(u;v)\ \text{for every}\ v\in\Sigma.$$

First, a simple estimation shows that $(h|_\Sigma)^0(u;v) \le h^0(u;v)$ for every $v \in \Sigma$. Thus, applying the above inequality for $v = Az \in \Sigma$ with $z \in X$ arbitrarily fixed, by (6.4) one has

$$\langle A^* w^*, z\rangle_X = \langle w^*, Az\rangle_\Sigma \le h^0(u; Az). \tag{6.5}$$

Using [9, Proposition 2.1.2 (b)] and (6.3), we get

$$\begin{aligned} h^0(u; Az) &= \max\{\langle x^*, Az\rangle_X : \ x^* \in \partial h(u)\} \\ &= \max\{\int_G \langle x^*, gz\rangle_X d\mu(g) : \ x^* \in \partial h(u)\} \\ &\le \int_G h^0(u; gz) d\mu(g) = \int_G h^0(g^{-1}u; z) d\mu(g) = \int_G h^0(u; z) d\mu(g) \\ &= h^0(u; z). \end{aligned}$$

Combining this relation with (6.5), we conclude that $A^* w^* \in \partial h(u)$. Since $w^* = A^* w^*|_\Sigma$, we obtain that $w^* \in \partial h(u)|_\Sigma$.

(b) Since $\partial h(u)$ is a nonempty, convex and weak*-compact subset of X^* (see [9, Proposition 2.1.2 (a)]), it is enough to prove that $\partial h(u)$ is G-invariant, i.e., $g\partial h(u) \subseteq \partial h(u)$ for every $g \in G$. To this end, let us fix $g \in G$ and $x^* \in \partial h(u)$. Then, for every $z \in X$ we have

$$\langle g x^*, z\rangle_X = \langle x^*, g^{-1}z\rangle_X \le h^0(u; g^{-1}z) = h^0(gu; z) = h^0(u; z),$$

i.e., $gx^* \in \partial h(u)$. □

Proof of Theorem 2.1. Let $u \in \Sigma$ be a critical point of $\mathcal{I}|_\Sigma$. Applying Proposition 6.1 one has $0 \in \partial(h|_\Sigma)(u) + \partial(\psi|_\Sigma)(u)$. Moreover, due to Lemma 6.1(a) we have

$$\emptyset \ne -\partial(h|_\Sigma)(u) \cap \partial(\psi|_\Sigma)(u) \subseteq -\partial h(u)|_\Sigma \cap \partial(\psi|_\Sigma)(u).$$

By choosing $K = \partial h(u)$ in Proposition 6.2 and taking into account Lemma 6.1(b), relation (6.1) implies that $\emptyset \ne -\partial h(u) \cap \partial\psi(u)$. Thus, in particular $0 \in \partial h(u) + \partial \psi(u)$, i.e., u is indeed a critical point of $\mathcal{I}$. □

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